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We will cover some of the following topics in the next week:

- derivatives in 2D space
- rates of change } ← focus for today
- Matlab code
- higher dimensions
- vectors → magnitude + direction
- vector fields
- partial derivatives
- dot products
- Vector projections
- gradients
- directional derivatives
- contour plots

Today we will focus on the first 4 topics, and tomorrow Alex will introduce vectors. Office hours and study group sessions will be held on Tuesday, Wednesday, Thursday, and Friday evening, and we encourage you to come to ask questions and work on the homework. In particular, Matlab, which is needed for the first assignment, is available in the Study room. Let's begin.

Rates of Change

We deal with rates of change all the time in our daily lives. Some examples include:

1. car speeds
2. hot dog eating speeds
3. velocity of a buoy in the ocean
4. heart rate
5. percent download rate
6. acceleration

Definition: A rate of change is a rate that describes how one quantity changes with respect to some other quantity

We can write y and x here. Then x is the independent variable and $y=f(x)$ is a function of x.

For the examples above, what is $y=f(x)$ and what is x?

1. $f(x)$ = position of car
   x = time
2. $f(x)$ = # of hot dogs eaten
   x = time

Exercise 1.1: Complete examples 3-6 on your own - determine $f(x)$ and x.

For these examples, x = time. But some rates of change are not taken with respect to time. For example, measurements of the incline or slope of a road are rates of change where $f(x)$ = height above ground and x = horizontal position on a road.

Let's do an example. Suppose I travel 50 miles to Boston in 2 hours (rush hour). What's my average speed (rate of change of position with respect to time)?

$$\text{average rate} = \frac{50 \text{ miles}}{2 \text{ hours}} = 25 \text{ mph}$$

But I probably didn't go 25 mph the whole time. Chances are I drove around 50 mph for most of the trip and then got stuck at 1 mph for the last hour downtown Boston. This is the difference between average rate of change and instantaneous rate of change.

Average rate of change in interval $[x_1, x_2] = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ measures how $f(x)$ changes with respect to x on average in the interval $[x_1, x_2]$

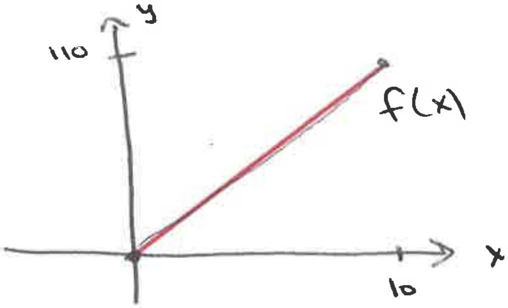
$\Longrightarrow$ In our last example, the average rate of change is 25 mph.

Instantaneous rate of change at a point x_1 measures how $f(x)$ is changing with respect to x at exactly that point.

$\Longrightarrow$ In our last example, this is the number on my speedometer at any time.

Let's do another example.

Example 1: Takeru Kobayashi set a world record in 2012 for eating 110 hot dogs in 10 minutes. Let's assume he ate them at a constant rate. Then the graph looks like this:



y = # hot dogs eaten

x = time in minutes

On average, what was Kobayashi's eating speed over the 10 minutes?

average rate of change in $[0,10] = \dfrac{f(10) - f(0)}{10 - 0} = 1.1$ HD/mi

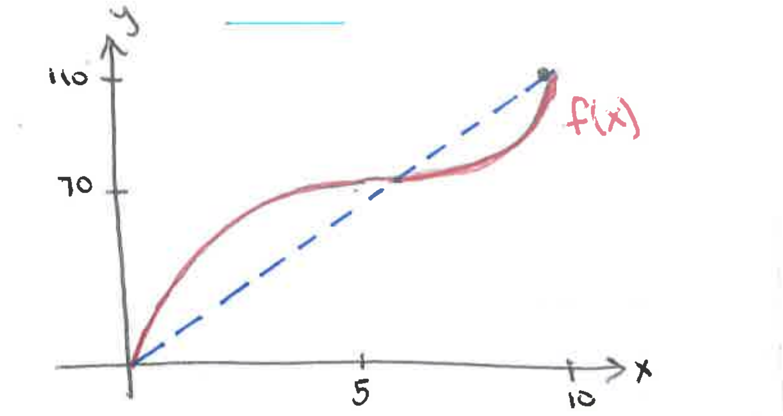
We assumed he ate at a constant speed. So what was his instantaneous eating rate at 2 minutes in? 1.1 HD/min. At 4 min? 1.1 HD/min. Constant rate.

We can also calculate the slope of the line we plotted.

$$\text{slope} = \frac{110 - 0}{10 - 0} = 1.1,$$

So the slope of the line is the same as the rate of change.

Example 2: Most likely, however, Kobayashi did not eat the hot dogs at a constant rate. Instead, he may have eaten really fast at first, then slowed down, took a break, and then finished strong. The graph might look like this:



What is the average rate of change in [0, 10] now? It's the same, 1.1 HD/min.

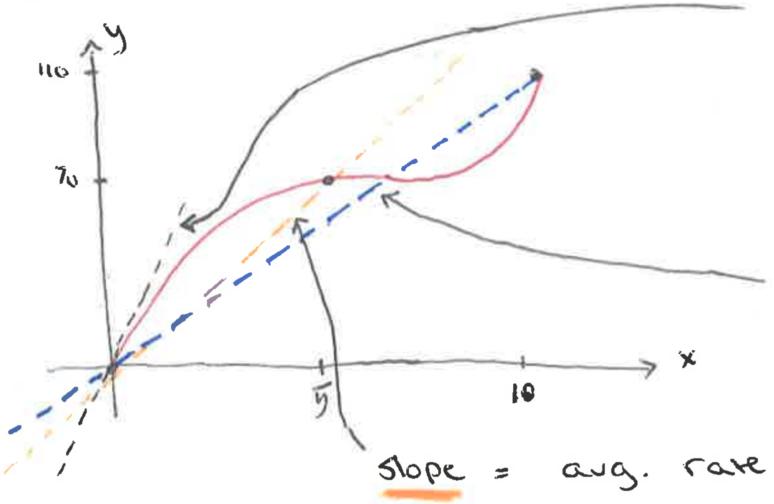
This rate is the same as the slope of the line connecting the endpoints $x=0$ and $x=10$. Suppose instead that we want to know what Kobayashi's initial eating speed was at $x=0$. Then it might make sense to average his speed over a shorter interval, say $x=0$ to 5.

Then

$$\text{avg. rate of change in } [0,5] = \frac{110-70}{5-0} = 8 \text{ HD/min.}$$

This is closer to the instantaneous rate at $x=0$ minutes in, but you can see it's not quite right.

Idea: To approximate the instantaneous rate of change, calculate the avg. rate of change over smaller and smaller intervals.

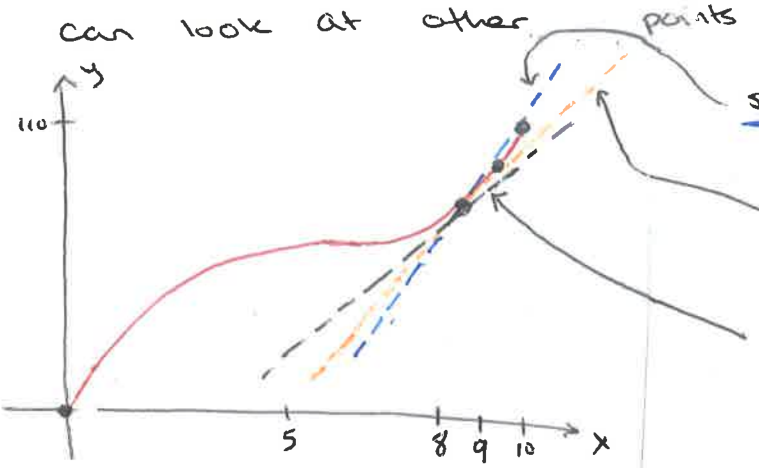


slope = average rate of change in [0, h] for h really small → almost instaneous slope of tangent line.

slope = average rate of change from [0, 10]

slope = avg. rate for $x=0$ to $x=5$.

We can look at other points on the curve, too.



slope = avg. rate of change on [8, 10]

slope = avg. rate of change on [8,9]

slope = avg. rate of change from 8 to 8+h.

(Note: lines connecting 2 points on the curve are called secant lines.)

Thus, what we've seen is that we can approximate the instantaneous rate of change at x=a as

$$\frac{f(a+h) - f(a)}{a+h - a} = \frac{f(a+h) - f(a)}{h}$$

as h gets really small.

This leads to the concept of derivative. The derivative of f(x) with respect to x at the point x=a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Sometimes we write $\frac{df}{dx}(a)$ instead of $f'(a)$. The derivative is

- an instantaneous rate of change
- the slope of the tangent line to the curve f(x) at the point x=a
- the limit of average rates over shorter and shorter intervals.

Let's calculate some derivatives.

Example 3: Let $f(x) = x^2$. What is the derivative at $x=3$?

Here $a=3$. Then

$$\frac{df}{dx}(3) = f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h}$$

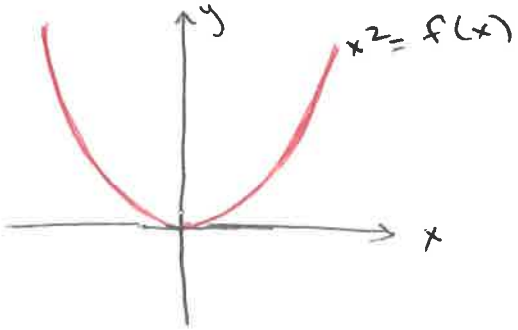
$$= \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h}$$

$$= \lim_{h\to 0} \frac{6h + h^2}{h}$$

$$= \lim_{h\to 0} 6 + h$$

$$= 6.$$

Now let's check is $f'(3)=6$ makes sense according to our intuition.



Let's approximate the average rate of change from $x=3$ to $x=5$:

- avg. rate = $\dfrac{f(5) - f(3)}{5-3} = \dfrac{25-9}{2} = 8$

Now from $x=3$ to $x=4$:

- avg. rate = $\dfrac{f(4) - f(3)}{4-3} = \dfrac{16-9}{1} = 7$

From $x=3$ to $x=3.5$:

- avg. rate = $\dfrac{12.25 - 9}{3.5 - 3} = 6.5$

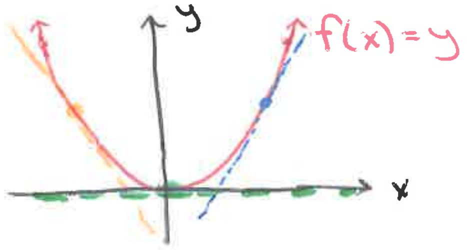
From $x=3$ to $x=3.1$:

- avg. rate = $\dfrac{9.61 - 9}{0.1} = 6.1$.

As you can see, the average rate of change approaches the derivative as the interval gets smaller and smaller, so our intuition checks out.

Example 4: See Matlab code.

In the examples we have chosen, the derivative is always positive, but it can also be negative or zero. Let's look at $f(x) = x^2$ again.



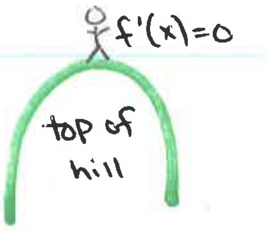
- Where is $f'(x)$ positive?
  → $f'(x)$ is positive when y is increasing as x increases
- Where is $f'(x)$ negative?
  → $f'(x)$ is negative when y is decreasing as x increases
- Where is $f'(x)$ zero?
  → This is a special case that represents a turning point (here at $x=0$) or that the function is not changing at that point.

Take-away:

$f'(x)>0$ → slope of tangent is positive (up hill)

$f'(x)<0$ → slope of tangent is negative (down hill)

$f'(x)=0$ → slope of tangent is zero (flat ground, top of hill, or bottom of valley)

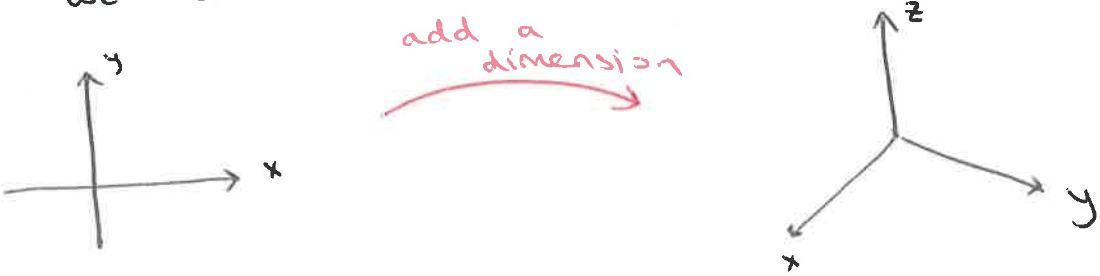


Exercise 1.2: Calculate the derivative of $f(x) = x^2 + 3x$ at $x=2$ using the definition that $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

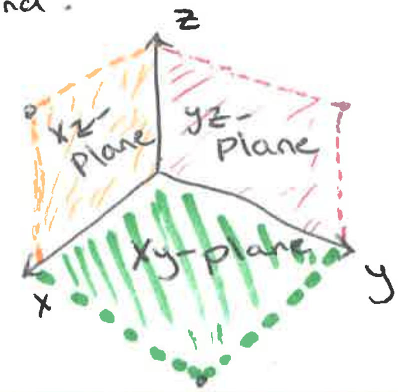
Higher dimensions

So far we have talked about functions of the form $y = f(x)$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$ are real numbers.

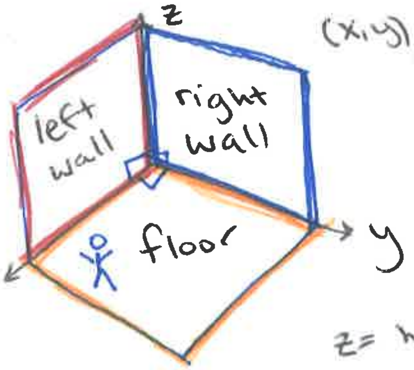
Now we will talk about 3D space.



Intuition: Think of x and y as telling you your position in the room and z as telling you how high above the floor you are when you jump off the ground.



Think:



(x,y) = position you are standing at before you jump

z= height of jump

Now we will change from looking at functions like $y = f(x)$ to studying functions of the form $z = f(x,y)$.

This means we now have 2 independent variables (your x,y position on the floor) and 1 dependent variable (height above ground).

Where would functions of the form $z = f(x,y)$ show up in real life?

1. Height above sea level

$x =$ latitude
$y =$ longitude
$z = f(x,y) =$ height above sea level

2. Temperature maps

$x =$ latitude
$y =$ longitude
$z = f(x,y) =$ temperature at point (x,y)

3. Density of trees in a forest

Exercise 1.4: Determine x, y, and z for Ex. 3 above.

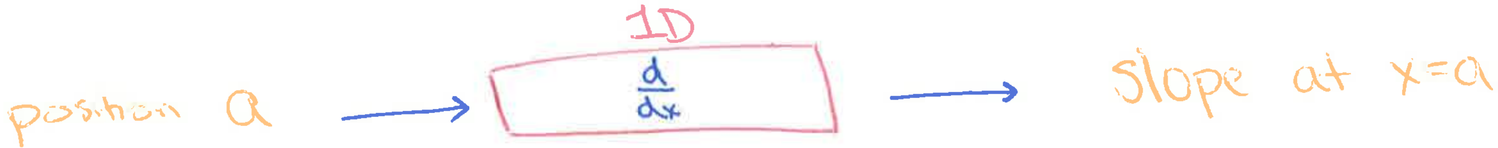
Goal: We'd like to be able to calculate rates of change and derivatives in 3D, and this will be the focus of the remainder of the class.

Recall for 2D, we said

$$\text{derivative of } f \text{ at } a = \text{instantaneous rate of change of } f \text{ at } a = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = f'(a).$$

Thus, $\frac{d}{dx}$ is a rule that takes a point and outputs a slope.

In other words, the derivative is an operator that takes a point and gives you a slope:



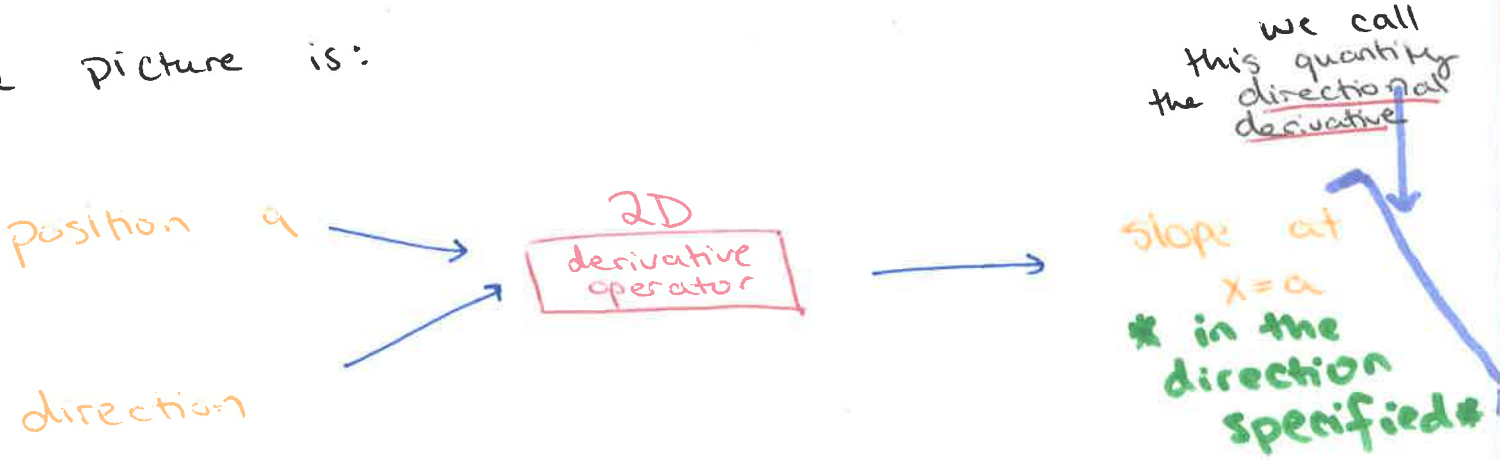
To get an idea for how this picture would change in 3D space, we will focus on one example. Our example will be the set-up below:

(x, y) = latitude & longitude position

z = height above sea level

We will be interested in one rate of change: the change in z or slope of ground.

If you take a walk around Brown, you will see that the slope of the ground depends on the direction you are looking – not just the point at which you are standing. Thus, in 3D space, the picture is:



It turns out this means we need to make one change in our initial definition of the derivative. Our new idea is:

$$\text{slope in 2D at the point } a = \lim_{h \to 0} \frac{f(a + \bar{u}h) - f(a)}{h}$$

or "directional derivative"

ne only change is the $\overline{u}$. This is what tells us ne direction, and it is called a vector. In order to study directional derivatives and slope in our 3D world, we need to learn more about vectors. That's our next topic.