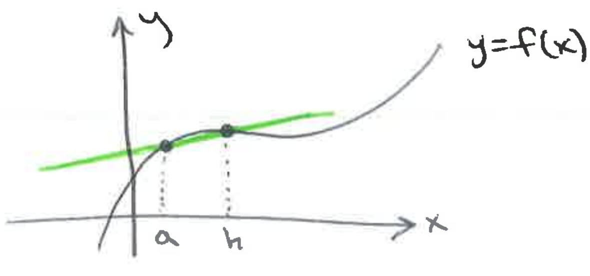
Let's begin by summarizing some of the key concepts we saw in the last lectures. Recall that:

1. Derivatives are a measure of the instantaneous rate of change at a point

2. In 2D, we consider functions of the form $y=f(x)$ and the derivative is the slope of the tangent line to $f(x)$ at the point $x=a$.

3. Intuitively, the derivative is the average rate of change over smaller and smaller intervals. We have:

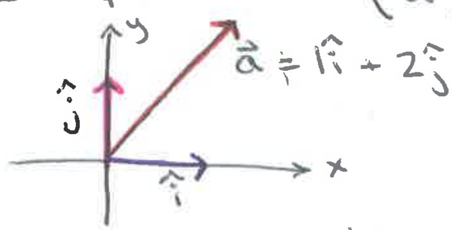
$$\frac{df}{dx}(a) = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

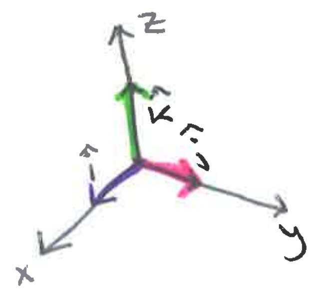


$h \to 0$

4. Vectors are quantities with magnitude + direction.

Example: $\vec{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\hat{i} + 2\hat{j}$





(The vectors $\hat{i}, \hat{j}, \hat{k}$ are length 1)

5. The length of a vector $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a}$

6. Unit vectors have length 1. The unit vector $\hat{a}$ in the direction of $\vec{a}$ is $\hat{a} = \vec{a} / \|\vec{a}\|$.

Our overarching goal for this course is:

Goal: Describes rates of change + derivatives in 3D space

Today we will cover the following topics:

see Matlab code plotCatalyst.m

- partial derivatives
- visualization of surfaces in 3D
- geometric intuition of partial derivatives
- gradient vectors
- vector fields
- contour plots

Motivation: Directional derivatives

Suppose you are walking around Brown carrying heavy bags in the heat. Then you'd be acutely aware of the fact that walking in the direction of your dorm meant going up hill, for example. In this case,

$(x,y) =$ latitude & longitude position

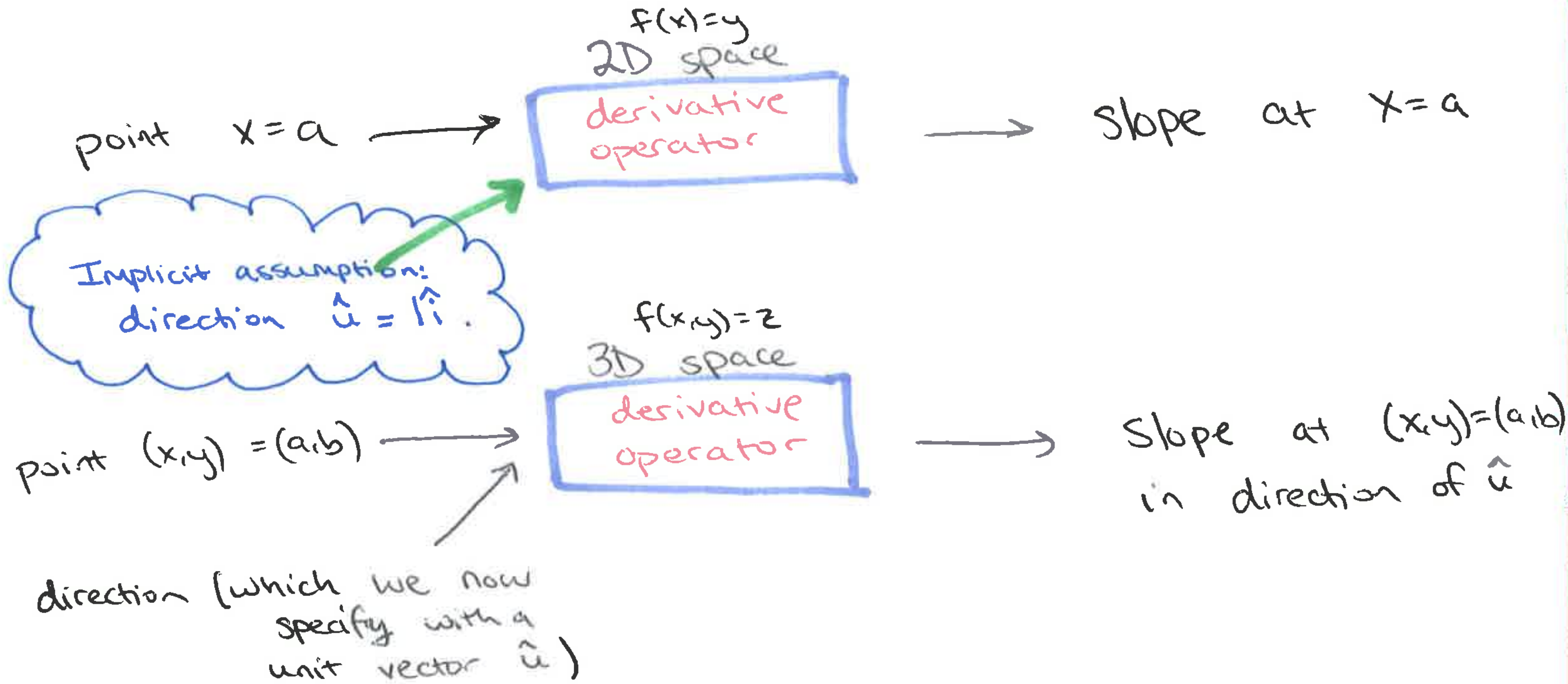
$z = f(x,y) =$ height above sea level

rate of interest = slope of ground in the direction you have to walk

Keep this example in mind as we talk about 3D space

As we mentioned in Lecture 1, we call the slope at any point (given the direction you are walking in) the directional derivative.

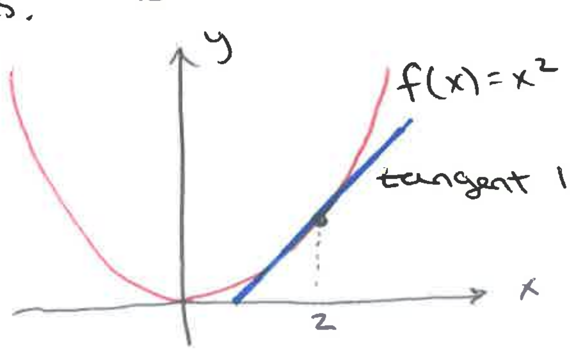
We drew these 2 pictures in lecture 1:



Before we move into today's new topics, I want to address a question several of you asked after lecture 1:

"Aren't there 2 directions to walk in for the case $f(x)=y$? So don't you need a direction specified in 2D space, too?"

This is a thoughtful question, and the answer is yes! To see this, consider $f(x)=x^2$:



tangent line at $x=2$ has slope 4

$\Rightarrow f'(2)=4$

The tangent line appears up hill (positive slope) when you walk in the positive x-direction $\longrightarrow$, but appears down hill (negative slope) when you walk in the negative x-direction $\leftarrow$. Given that $f'(2) = 4$, what is the slope of the line when you walk to the left? It has the same magnitude, but a different direction, so the rate of change of $f(x)$ in the negative x-direction is -4.

Take-away: when we speak of the derivative in 2D space for $f(x) = y$, it is always assumed that we mean the rate of change of $f(x)$ with respect to the positive x-direction.

How can we express this using vectors? This is just the direction specified by $\hat{u} = 1\hat{i}$! Similarly, if we are interested in how $f(x)$ changes in the negative x-direction, we want the directional derivative at a point $x = a$ in the direction $\hat{u} = -1\hat{i}$.

We can update our picture on the last page to recognize our implicit assumption that $\hat{u} = 1\hat{i}$, but it should be kept in mind that it is standard process to take for granted that $f'(x)$ is always the directional derivative in the direction $\hat{u} = 1\hat{i}$, and we will speak of it as just the derivative.

Important Difference between 2D & 3D:

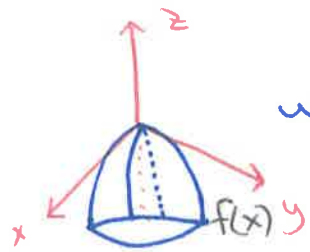
In 2D, we could walk in 2 directions: $\hat{u} = 1\hat{i}$ or $\hat{u} = -1\hat{i}$, and the resulting 2 slopes differed only by a sign. In 3D, there are infinitely many directions you can walk in, and the resulting slopes can be very different: for example, the ground may be level (slope = 0) in one direction but down hill in another.

We'll talk more about directional derivatives in the remaining lectures. Our next topic is partial derivatives ☺.

Aside: Check out the Matlab code plotCatalyst.m to visualize surfaces $z = f(x,y)$ in 3D space.

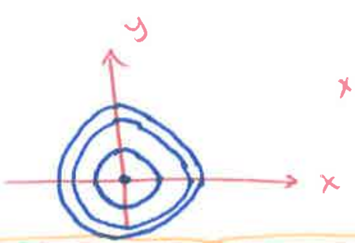
Contour plots (also called level curves) are what you get when you draw curves given by $f(x,y) = c$ in the xy-plane. You can think of these as horizontal slices across your mountain. For example:

$f(x,y) = -x^2 - y^2$

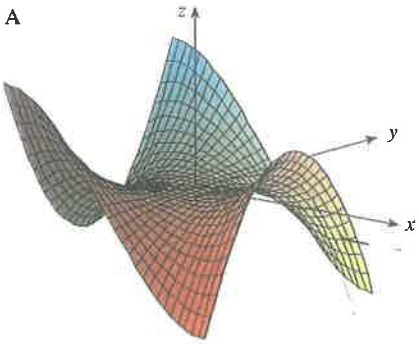


upside down bowl

Contour plot:



Aside: Practice with Contour Plots
Match the 3D surface $z=f(x,y)$ to its contour plot in the xy-plane

A


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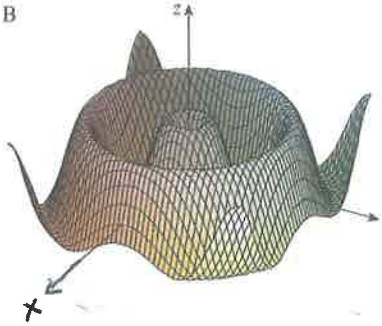

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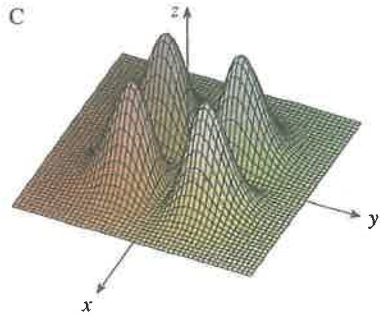

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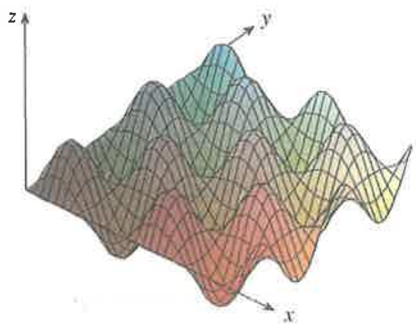

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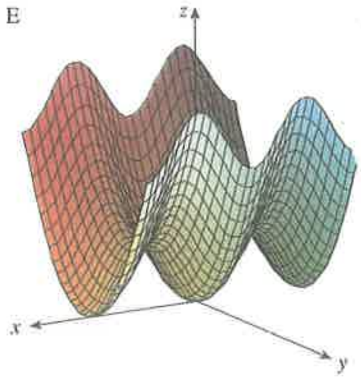

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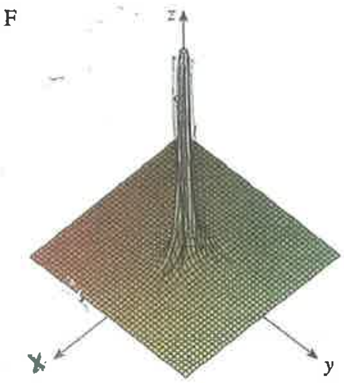

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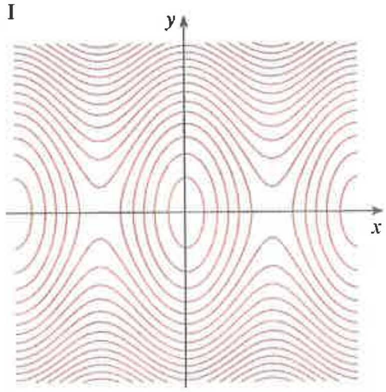

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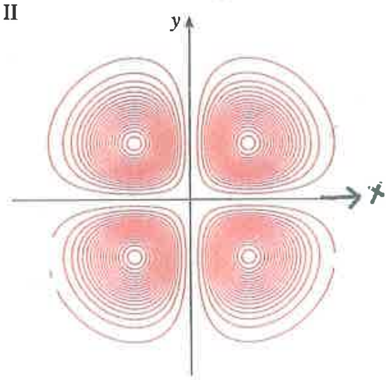

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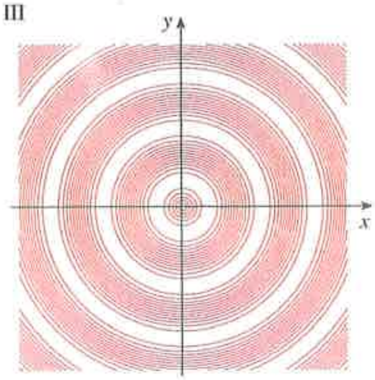

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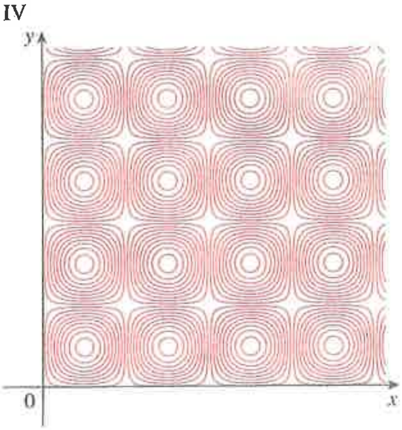

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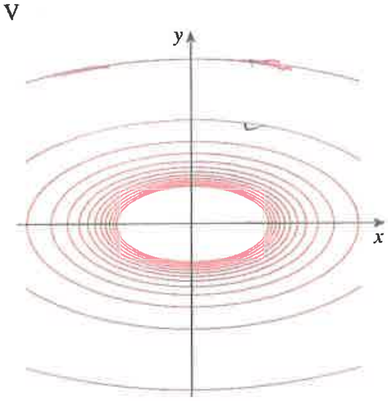

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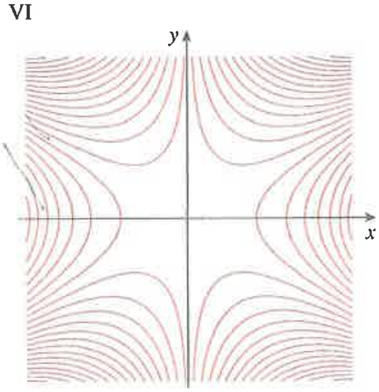


Partial Derivatives

Let's start our study of rates of change for functions of the form $z = f(x,y)$ with an example.

Example: Heat + Humidity

As you probably have recognized, days with higher humidity feel hotter than days with lower humidity, despite the thermostat displaying the same temperature. To better account for this, we sometimes speak of the heat index – this is the temperature it feels like outside, and it depends on actual temperature and humidity. Let's write

x = relative humidity

y = actual temperature

$z = f(x,y)$ = heat index (perceived temp)

The National Weather Service lists the following relationships between x, y, and z:

Relative humidity x%

Actual temp y (°F)

y \ x	50	55	60	65	70	75
90	96	98	100	103	106	109
92	100	103	105	108	112	115
94	104	107	111	114	118	122
96	109	113	116	121	125	130
98	114	118	~~129~~ 123	~~135~~ 127	133	138

We will focus on understanding how $z = f(x,y)$ changes at the point $(x,y) = (70, 96)$, and we will do this in 2 ways:

Part 1:

Let's set $y = 96°F$ and focus on how z changes when we change the humidity x. This means we are looking at the second to last row in our table (highlighted in yellow). This row tells us $z = f(x, 96)$ → since we set $y = 96°F$, there is only one independent variable. We want to know the rate of change of z with respect to x when $y = 96°F$. To do this and shorten our notation, we can write $g(x) = f(x, 96)$. Then

$x =$ humidity
$g(x) =$ heat index z when $y = 96°F$.

But now we know how to find the rate of change of $g(x)$ with respect to x! It's just the derivative:

$$g'(a) = \frac{dg}{dx}(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

We said we wanted to know the rate of change at the point $(x,y) = (70, 96)$, so $a = 70$ and

$$g'(70) = \lim_{h \to 0} \frac{g(70+h) - g(70)}{h}$$

(using the values from our table) →
$$\approx \frac{g(75) - g(70)}{5} = \frac{f(75,96) - f(70,96)}{5}$$

$$= \frac{130 - 125}{5} = 1$$

According to our rough approximation, $\frac{dg}{dx}(70) \approx 1$.
This means that: when the actual temp is 96°F and the humidity is 70%, the heat index goes up by ≈ 1°F every time the humidity rises by 1%!

Part 2:

Now let's set $x = 70\%$ humidity constant and focus on how z changes when we change the temperature y.
We are now looking at the second to last column in our table (highlighted in yellow). This column tells us $z = f(70, y)$ → since we set $x = 70\%$, we have only one independent variable left. Thus, let $m(y) = f(70, y)$.
Then

y = temperature

$m(y)$ = heat index z when $x = 70\%$ humidity.

We know how to find the derivative of m with respect to y then:

$$m'(b) = \frac{dm}{dy}(b) = \lim_{h \to 0} \frac{m(b+h) - m(b)}{h}$$

We wanted to know the rate of change at the point $(x, y) = (70, 96)$, so $b = 96$ and

$$m'(96) = \frac{dm}{dy}(96) = \lim_{h \to 0} \frac{m(96+h) - m(96)}{h}$$

$$\approx \frac{m(98) - m(96)}{2} = \frac{f(70, 98) - f(70, 96)}{2}$$

$$= \frac{133 - 125}{2} = 3.5$$

According to our rough approximation, $\frac{dm}{dy} \approx 3.5$. This

Means that: when the actual temp is 96°F and the humidity is 70%, the heat index goes up by ≈ 3.5°F every time the temp rises 1° !

Conclusions: What did the heat index example show?
we found that, at $x = 70\%$ humidity and $y = 96°F$,

1. the heat index $z = f(x,y)$ changes at a rate of 1°F per humidity change of 1%
2. the heat index $z = f(x,y)$ changes at a rate of 3.5°F per temperature change of 1°.

Take-away: For functions of the form $z = f(x,y)$, there are 2 natural questions we can ask about rates of change:

1. Given a point $(x,y) = (a,b)$, what is the rate of change of $f(x,y)$ with respect to x?

   → we will call this $\frac{df}{dx}(a,b)$, the partial derivative of $f(x,y)$ with respect to x at the point (a,b).

2. Given a point $(x,y) = (a,b)$, what is the rate of change of $f(x,y)$ with respect to y?

   → we will call this $\frac{df}{dy}(a,b)$, the partial derivative of $f(x,y)$ with respect to y at the point (a,b).

In our example, $g'(a) = \frac{df}{dx}(a,b)$ told us how $f(x,y)$ changes with x and $m'(b) = \frac{df}{dy}(a,b)$ told us how $f(x,y)$ changes with y.

We saw that

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}.$$

Using that $g(a) = f(a,b)$, we can rewrite this as:

$$\frac{df}{dx}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

And this leads to a definition:

For a function of the form $z = f(x,y)$, its partial derivatives $\frac{df}{dx}(a,b)$ and $\frac{df}{dy}(a,b)$ at the point $(x,y) = (a,b)$ are given by

$$\frac{df}{dx}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\frac{df}{dy}(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

Partial derivatives, just like regular derivatives, are rates of change. $\frac{df}{dx}$ tells you the rate of change of f with respect to x and $\frac{df}{dy}$ tells you the rate of change of f with respect to y.

Rules for finding Partial derivatives of $z = f(x,y)$:

1. To find $\frac{df}{dx}(a,b)$, set $y=b$ constant and differentiate with respect to x (think: $g(x) = f(x,b)$ and find $g'(a) = \frac{df}{dx}(a,b)$)

2. To find $\frac{df}{dy}(a,b)$, set $x=a$ constant and differentiate with respect to y (think: $m(y) = f(a,y)$ and find $m'(b) = \frac{df}{dy}(a,b)$).

Let's do an example.

Example 2: Find the partial derivatives of $f(x,y) = x^2 + y^2$ at the point $(a,b) = (1,2)$.

We begin by finding $\frac{df}{dx}(1,2)$:

$$\frac{df}{dx}(1,2) = \lim_{h \to 0} \frac{f(1+h,2) - f(1,2)}{h}$$

$$= \lim_{h \to 0} \frac{((1+h)^2 + 2^2) - (1^2 + 2^2)}{h}$$

$$= \lim_{h \to 0} \frac{1 + h^2 + 2h + 4 - 1 - 4}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2h}{h} = \lim_{h \to 0} h + 2 = 2.$$

Now we find $\frac{df}{dy}(1,2)$:

$$\frac{df}{dy}(1,2) = \lim_{h \to 0} \frac{f(1, 2+h) - f(1,2)}{h}$$

$$= \lim_{h \to 0} \frac{(1^2 + (2+h)^2) - (1^2 + 2^2)}{h}$$

$$= \lim_{h \to 0} \frac{1 + 4 + 4h + h^2 - 1 - 4}{h}$$

$$= \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} 4 + h = 4.$$

We conclude that $\frac{df}{dx}(1,2) = 2$ and $\frac{df}{dy}(1,2) = 4$.

Exercise 3.1: Using the definition of partial derivatives, find the rate of change of $z = f(x,y)$ with respect to y at the point $(a,b) = (1,1)$ for $f(x,y) = 4 - x - 2y^2$.

Remark: You actually calculated a partial derivative in the first HW assignment! Recall that we asked you to find $f'(3)$ for $f(x) = cx$. If we rewrite this as $g(x,c) = cx$, we can express this as a function in 3D! Then what you did was assume c was constant and differentiate w.r.t. (with respect to) x. This is the same as finding $\frac{dg}{dx}(3,c)$!

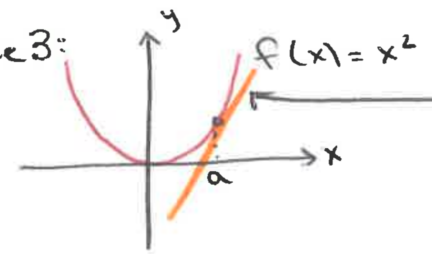
Exercise 3.2: Rewrite $f(x) = cx$ as $g(x,c) = cx$. Find $\frac{dg}{dx}(3,2)$.

Our next section focuses on connecting partial derivatives to our intuition from 2D space that derivatives are the same as slopes of tangent lines.

Tangent lines:

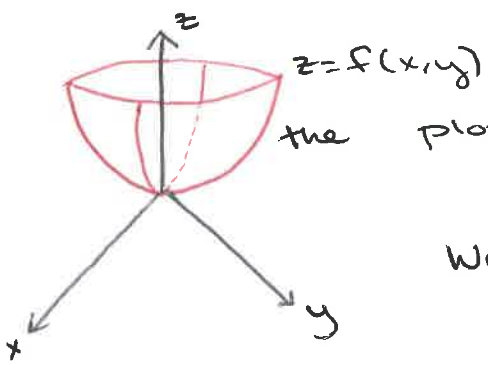
Recall that in 2D, for functions of the form $y=f(x)$, the derivative is the slope of the tangent line to the function at the point x.

Example 3:



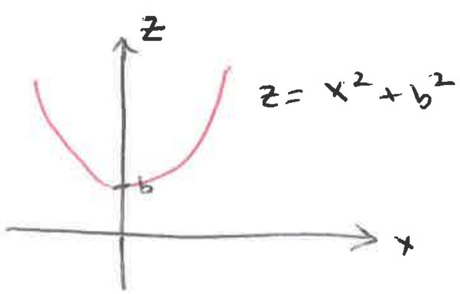
slope of tangent line $= f'(a)$

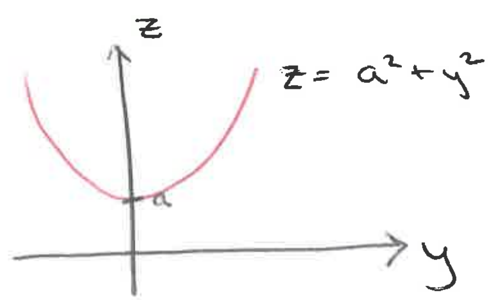
Is there something similar going on in 3D? It turns out yes! Let's plot the function $z=f(x,y)=x^2+y^2$.



the plot looks like a bowl

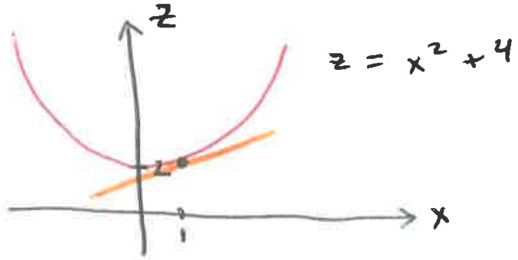
We can also plot $z=f(x,b)=x^2+b^2$ and $z=f(a,y)=a^2+y^2$:





These are just plots in 2D space. From Example 2, we know $\frac{df}{dx}(1,2)=2$. This is the same as asking what the derivative of $z=x^2+4$ is at $x=1$!

We know how to find that:



It's the slope of the tangent line to the curve $z = x^2 + 4$ at the point $x = 1$!

Take-away: Partial derivatives are related to tangent lines, too! For a function $z = f(x,y)$, $\frac{df}{dx}(a,b)$ is the slope of the tangent line to the curve $z = f(x,b) = g(x)$ at the point $x = a$ in 2D space.

Gradient Vectors

Now we will define a special vector called the gradient vector (or just "gradient" for short). The gradient is just a short way of combining $\frac{df}{dx}$ and $\frac{df}{dy}$, but in the future we will see that the gradient actually stores all the info about rates of change of $z = f(x,y)$.

In this class we will consider gradient vectors associated with the function $z = f(x,y)$. Then we define:

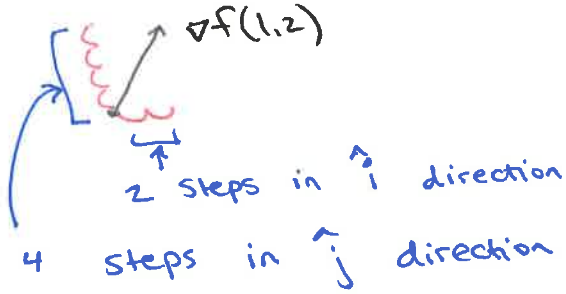
$$\text{gradient of } f = \nabla f(a,b) = \begin{pmatrix} \frac{df}{dx}(a,b) \\ \frac{df}{dy}(a,b) \end{pmatrix} = \frac{df}{dx}(a,b)\,\hat{i} + \frac{df}{dy}(a,b)\,\hat{j}$$

(or just ∇f for short)

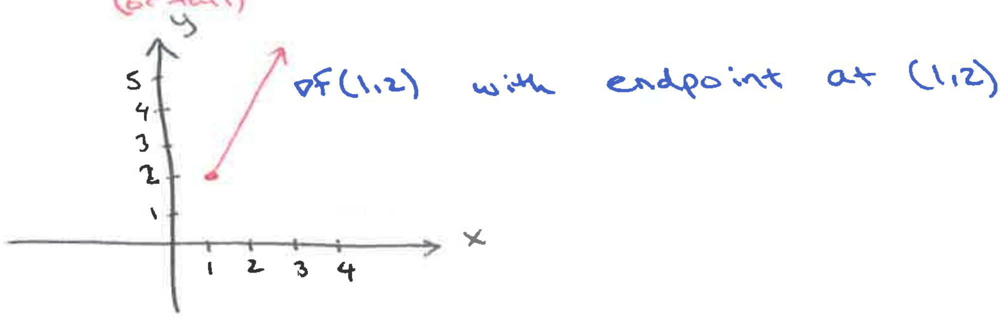
Example 4:

Using the work we did in Example 2 in these notes (finding that for $f(x,y) = x^2 + y^2$ we have $\frac{df}{dx}(1,2) = 2$ and $\frac{df}{dy}(1,2) = 4$), we can find the gradient:

$$\nabla f(1,2) = 2\hat{i} + 4\hat{j}$$



It is often useful to draw gradient vectors with their endpoint (or tail) located at (a,b). Then we have:



$\boxed{\text{Vector Fields}}$ Rather than just drawing $\nabla f(a,b)$ at one point (a,b), we can draw it at a collection of points on a grid. This is called a gradient vector field.

You will have more practice computing gradients in the homework.

Before we conclude for today, let's state the definition of the directional derivative of $z = f(x,y)$ at the point (a,b) in the direction $\hat{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

$$\text{directional derivative } D_u f(a,b) = \lim_{h \to 0} \frac{f(a + u_1 h,\, b + u_2 h) - f(a,b)}{h}$$

Recall that we found 2 partial derivatives today:

$$\frac{df}{dx}(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

and

$$\frac{df}{dy}(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

Do you see any similarities with ✱? $\frac{df}{dx}(a,b)$ is the same as the directional derivative $D_u f(a,b)$ when $\hat{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$! Thus, the partial derivatives are just the directional derivatives in the directions $\hat{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively!

As you know, however, there are more than just 2 directions to walk in in 3D space (and more than 2 tangent lines you can draw to a 3D surface — in fact, you can draw a whole plane of tangent lines!), so we have a lot more directional derivatives we could calculate. It turns out ∇f, the gradient, houses all the info

about directional derivatives we could wish for.
In other words, the gradient vector can be used to tell us the rate of change of $z = f(x,y)$ in any direction! In order to get this info out of the gradient, we need 2 more tools: vector projections and dot products.