1 Review

Definition 1.1. A sequence $u_m \in V$ is a *Palais-Smale sequence* for E if $|E(u_m)| \leq c$ uniformly in m, while $||DE(u_m)|| \to 0$ as $m \to \infty$. (Struwe)

Definition 1.2. E satisfies the *Palais-Smale condition* (P-S) if any Palais-Smale sequence has a strongly convergent subsequence. (Struwe)

Theorem 1.3. *Minimax Principle: Supoose M is a $C^{1,1}$ Finsler manifold and $E \in C^1(M)$ satisfies P-S. Also suppose $\mathcal{F} \subset \mathcal{P}(M)$ is a collection of sets which is invariant with respect to any continuous semi-flow $\Phi : M \times [0, \infty) \to M$ such that $\Phi(\cdot, 0) = id$, $\Phi(\cdot, t)$ is a homeomorphism of M for any $t \geq 0$, and $E(\Phi(u, t))$ is non-increasing in t $\forall u \in M$. Then if*

$$\beta = \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u)$$

is finite, β is a critical value of E. (Struwe)

Theorem 1.4. *Suppose M is a complete $C^{1,1}$ Finsler manifold and $E \in C^1(M)$ satisfies P-S. Let $\beta \in \mathbb{R}, \bar{\epsilon} > 0$ be given and let N be any neighborhood of K_β. Then $\exists \epsilon \in (0, \bar{\epsilon})$ and a continuous 1-parameter family of homeomorphisms $\Phi(\cdot, t)$ of M, $0 \leq t < \infty$, with the properties*

1. *$\Phi(u, t) = u$ if $t = 0$, or $DE(u) = 0$, or $|E(u) - \beta| \geq \bar{\epsilon}$.*
2. *$E(\Phi(u, t))$ is non-increasing in t for any $u \in M$.*
3. *$\Phi(E_{\beta+\epsilon} \setminus N, 1) \subset E_{\beta-\epsilon}$ and $\Phi(E_{\beta+\epsilon}, 1) \subset E_{\beta-\epsilon} \cup N$.*

If M admits a compact symmetry group G and E is $G-$invariant, then Φ can be constructed so that $\Phi(g(u), t) = g(\Phi(u, t))$. (Struwe)

2 Krasnoelskii Genus

While working with the Krasnoelskii genus, our set-up will always be:

V is a Banach space.
M is a closed symmetric $C^{1,1}$ submanifold of V.
E is an even C^1 functional on M and satisfies P-S.
$\mathcal{A} = \{A \subset V | A \text{ closed }, A = -A\}$.
h is any odd continuous map.
$E_\beta = \{u \in M | E(u) < \beta\}$.
$K_\beta = \{u \in V | E(u) = \beta,\, DE(u) = 0\}$.

Definition 2.1. For $A \in \mathcal{A}$ with $A \neq \emptyset$, let

$$\gamma(A) = \begin{cases} \inf\{m | \exists h \in C^0(A; \mathbb{R}^m \setminus \{0\}), h(-u) = -h(u)\} \\ \infty, \text{ if } \{...\} = \emptyset, \text{ in particular, if } 0 \in A, \end{cases}$$

and define $\gamma(\emptyset) = 0$. We call $\gamma(A)$ the *Krasnoselskii genus* of A. (Struwe)

Proposition 2.2. *Let $A, A_1, A_2 \in \mathcal{A}$ and $h \in C^0(V;V)$ odd. Then:*

1. $\gamma(A) \geq 0$ *and* $\gamma(A) = 0$ *iff* $A = \emptyset$.
2. $A_1 \subset A_2$ *iff* $\gamma(A_1) \leq \gamma(A_2)$.
3. $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$.
4. $\gamma(A) \leq \gamma(\overline{h(A)})$.
5. *If* $A \in \mathcal{A}$ *is compact and* $0 \neq A$, *then* $\gamma(A) < \infty$ *and* $\exists$ *a neighborhood* N *of* A *in* V *such that* $\overline{N} \in \mathcal{A}$ *and* $\gamma(A) = \gamma(\overline{N})$. *(Struwe)*

Lemma 2.3. *With the set-up given above, suppose for* k, ℓ *there holds*

$$-\infty < \beta_k = \beta_{k+1} = ... = \beta_{k+\ell-1} = \beta < \infty,$$

where $k \leq \gamma(M)$. *Then* $\gamma(K_\beta) \geq \ell$. *In particular, if* $\ell > 0$, *then* K_β *is infinite.*

Theorem 2.4. *Given the same set-up above, suppose in addition that* E *is bounded from below on* M. *Let* $\hat{\gamma}(M) = \sup\{\gamma(K) | K \subset M \text{ compact, symmetric}\}$. *Then* E *has at least* $\hat{\gamma}(M) \leq \infty$ *pairs of distinct critical points. (Struwe & Costa)*

3 General Index Theory

For the case of general index theory, our set-up is:

M is a complete $C^{1,1}$ Finsler manifold with a compact group action G.
$\mathcal{A} = \{A \subset V | A \text{ closed, } g(A) = A \text{ for all } g \in G\}$.
$\Gamma = \{h \in C^0(M;M) | h \circ g = g \circ h \text{ for all } g \in G\}$.
$\text{Fix } G = \{u \in M | gu = u \text{ for all } g \in G\}$.

Definition 3.1. An index for $(G, \mathcal{A}, \Gamma)$ is a mapping $i : \mathcal{A} \to \mathbb{N}_0 \cup \{\infty\}$ such that for all $A, B \in \mathcal{A}$ and $h \in \Gamma$ there holds

1. $i(A) \geq 0$ and $i(A) = 0$ iff $A = \emptyset$.
2. $A \subset B$ iff $i(A) \leq i(B)$.
3. $i(A \cup B) \leq i(A) + i(B)$.
4. $i(A) \leq i(\overline{h(A)})$.
5. If $A \in \mathcal{A}$ is compact and $A \cap \text{ Fix } G = \emptyset$, then $i(A) < \infty$ and $\exists$ a $G-$invariant neighborhood N of A such that $i(\overline{N}) = i(A)$.
6. If $u \neq \text{ Fix } G$, then $i(\cup_{g \in G} gu) = 1$. (Struwe)

Theorem 3.2. *With the set-up given above, suppose the functional* $E \in C^1(M)$ *is bounded from below and satisfies P-S. Suppose* G *acts on* M *without fixed points and let* i *be an index for* $(G, \mathcal{A}, \Gamma)$. *Define* $\hat{i}(M) = \sup\{i(K) | K \subset M \text{ with } K \text{ compact and } G-\text{invariant}\} \leq \infty$. *Then* E *admits at least* $\hat{i}(M)$ *critical points which are distinct modulo* G. *(Struwe)*

4 References

- *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems* by M. Struwe
- *An Invitation to Variational Methods in Differential Equations* by D. Costa.